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Lengths of analytic well-orderings

by

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1. Introduction. This paper is a continuation of our [6]. Here we shall consider lengths of Σ_{2n+1}^{1} well-orderings of natural numbers as well as those of Π_{2n}^{1} under the assumption of the axiom of projective determinateness (*APD*).

Let K be a class of predicates and/or sets. A relation $R(x, y)$ on natural numbers is a K well-ordering (of natural numbers) iff R belongs to the class K and $\lambda xyR(x, y)$ well-orders its field $F(R)=\{x\mid(\exists z)[R(x, z)\vee R(z, x)]\}$. A countable ordinal ν is called a K-ordinal if ν is the order type of a K well-ordering of natural numbers. Let $\omega(K)$ be the first non K-ordinal. In [6], we considered $\omega(\Delta_{k}^{1})$, $\omega(\Pi_{k}^{1})$ and $\omega(\Sigma_{k}^{1})$, and compared their size under the assumption of the axiom of constructibility $V=L$ (Gödel [2]) for $k>2$; thus

$$(1)\qquad \omega(\Delta_{k}^{1}) = \omega(\Pi_{k}^{1}) < \omega(\Sigma_{k}^{1}) \quad \text{for } k>1\ .$$

There we asked whether or not (1) holds without the assumption $V=L$. However to settle this problem for $k>2$ seems to be impossible unless we assume another additional axiom than *ZF* (the axioms of Zermelo-Fraenkel set theory). Addison-Moschovakis [1] has shown that *APD* implies $\omega(\Delta_{k}^{1})<\omega(E_{k}^{1})$ for $k>0$, where E_{k}^{1} is Σ_{k}^{1} or Π_{k}^{1} according as k is even or odd. (See Appendix of the present paper.)

In what follows we shall show the following theorem. Let *APD* be the statement asserting that every projective game is determined, and let $\Upsilon_{2n}^{1}=\Pi_{2n}^{1}$ and $\Upsilon_{2n+1}^{1}=\Sigma_{2n+1}^{1}$. Then

THEOREM. *APD implies* $\omega(\Delta_{k}^{1})=\omega(\Upsilon_{k}^{1})$ *for each* $k>0$.[1)]

Of course, for $k=1, 2$ we do not need any additional axiom other than *ZF* ([6]). However I do not know whether or not the following conjecture proposed by *F*. Gass (private communication) is true: *ZF* implies $\omega(\Sigma_{k}^{1})\neq\omega(\Pi_{k}^{1})$ and $\omega(\Delta_{k}^{1})=\min(\omega(\Sigma_{k}^{1}), \omega(\Pi_{k}^{1}))$ for all $k>0$.

2. THEOREM 1. *APD implies* $\omega(\Delta_{k}^{1})=\omega(\Sigma_{k}^{1})$ *for each odd* k.

Proof. We illustrate this for $k=3$. Proof is analogous for the other cases. Let $R(x, y)$ be a relation satisfying the following con-

1) This result was also independently obtained by A. S. Kechris (private communication).

ditions:

(i) R is a Σ_3^1 well-ordering,

(ii) Every proper initial segment of R (i.e., for every $y \in F(R)$ $R_y = \{x \mid R(x, y) \wedge x \neq y\}$) represents a Δ_3^1-ordinal,

(iii) For every Δ_3^1-ordinal ν there exists a proper initial segment R_y such that $|R_y| = \nu$, where $|R_y|$ is the order type of R_y.

We shall show that this together with APD yields a contradiction. So we always assume APD and hence we can use results in Addison-Moschovakis [1] for $k=3$. We also borrow the notations W_k ($\subseteq N^2$) and G_w ($\subseteq N$) from there, where $k > 0$ and N is the set of all natural numbers. They are E_k^1 and Δ_k^1 (for $w \in W_k$), respectively. By (ii) and [1] we have

$$(1) \qquad (\forall y)(\exists w)(\exists e)[y \in F(R) \rightarrow w \in W_3 \wedge [\{e\}^{G_w} \text{ is a total function}] \wedge \mathrm{Bord}(\{e\}^{G_w}) \wedge |R_y| < |\{e\}^{G_w}|] ,$$

where for a number-theoretic function f Bord(f) means Bord($\lambda xy[f(\langle x, y\rangle) = 0]$), and for a 2-place relation A of natural numbers

$$\begin{aligned} \mathrm{Bord}(A) \leftrightarrow & (\forall x)(\forall y)[x, y \in F(A) \rightarrow A(x, y) \vee A(y, x)] \\ & \wedge (\forall x)(\forall y)[A(x, y) \wedge A(y, x) \rightarrow x = y] \\ & \wedge (\forall x)(\forall y)(\forall z)[A(x, y) \wedge A(y, z) \rightarrow A(x, z)] \\ & \wedge (\forall \alpha \in N^N)(\exists x)[\neg A(\alpha(x+1), \alpha(x)) \vee \alpha(x+1) = \alpha(x)] , \end{aligned}$$

and if Bord(f) $|f|$ denotes the order type of $\lambda xy[f(\langle x, y\rangle) = 0]$. Here $\langle x, y\rangle = 2^x \cdot 3^y$. Under the conditions $y \in F(R)$ and $w \in W_3$, Bord($\{e\}^{G_w}$) is Δ_3^1 ([6; Footnote 4)]), and $|R_y| < |\{e\}^w|$ is a Π_3^1 relation since both R_y, $\{e\}^{G_w}$ are well-orderings. Hence the bracketed predicate in (1) is E_3^1. By [1; VIII] there exists Δ_3^1 functions f and g such that

$$(\forall y)[y \in F(R) \rightarrow f(y) \in W_3 \wedge [\{g(y)\}^{G_{f(y)}} \text{ is a total function}] \wedge \mathrm{Bord}(\{g(y)\}^{G_{f(y)}}) \wedge |R_y| < |\{g(y)\}^{G_{f(y)}}|] .$$

Let $Q_y(\langle a, b\rangle) \leftrightarrow \{g(y)\}^{G_{f(y)}}(\langle a, b\rangle) = 0$ for $y \in F(R)$. Then Q_y is Δ_3^1 uniformly in $y \in F(R)$. So by [1; (V)] there exists a recursive function h such that Q_y is recursive in $G_{h(y)}$. Then we have

$$(2) \qquad (\forall w \in W_3)(\exists y \in F(R))[G_w \leq_T G_{h(y)}] .$$

where $G \leq_T G'$ means G is Turing reducible to G'. For, suppose there would be a w in W_3 such that for any $y \in F(R)$, $G_w \not\leq_T G_{h(y)}$. Since G_w and $G_{h(y)}$ are $\leq_T$-comparable, it must be

$$(\forall y \in F(R))[G_{g(y)} \leq_T G_w] .$$

Therefore for each $y \in F(R)$, Q_y is a recursive-in-G_w well-ordering. Hence by (iii)

$$\omega(\Delta_3^1) \leq \omega_1^{G_w} ,$$

where for each set A of natural numbers ω_1^A is the first non recursive-in-A ordinal. But since G_w itself is Δ_3^1, $\omega_1^{G_w}$ is also a Δ_3^1-ordinal. (Note that W^A, Spector [5], is Δ_3^1 if A is a Δ_3^1 set of natural numbers [6; Footnote 4)]. Also see the proof method in Appendix 1).) This is a contradiction, and hence (2) holds.

Now by (2) we shall show for $\alpha \in N^N$

$$(3) \qquad \alpha \in \Delta_3^1 \leftrightarrow (\exists y \in F(R))[\alpha \leq_T G_{h(y)}] .$$

The reverse implication of (3) is trivial. To prove ($\rightarrow$), let $\alpha \in \Delta_3^1$. Then by [1; V] there is a w in W_3 such that $\alpha \leq_T G_w$. For this w, by (2) we take a $y \in F(R)$ such that $G_w \leq_T G_{h(y)}$. Then $\alpha \leq_T G_{h(y)}$; and hence (3) holds.

By (3), $\Delta_3^1 \cap N^N$ is Σ_3^1, i.e. it is Υ_3^1 which contradicts the fact that for any odd k, $\Delta_k^1 \cap N^N$ is not Υ_k^1 [1; IX]. Therefore there is no R possessing the properties (i)-(iii).

Now if $\omega(\Delta_3^1) < \omega(\Sigma_3^1)$, then clearly there is such an R. So we have $\omega(\Delta_3^1) \not< \omega(\Sigma_3^1)$, which gives Theorem 1.

3. THEOREM 2. *APD implies* $\omega(\Delta_k^1) = \omega(\Pi_k^1)$ *for each even* $k > 0$.

Proof. By Moschovakis [4; Theorem 2] (see Kechris-Moschovakis [3; p. 31]), if *APD* holds, then every Σ_{2n}^1 subset of N^N contains a Δ_{2n}^1 element. Now let $R(x, y)$ be an arbitrary Π_{2n}^1 well-ordering. Then we have

$$(1) \qquad (\exists \alpha)[\mathrm{Bord}(\alpha) \wedge (\forall x)(\forall y)\{R(x, y) \rightarrow \alpha(\langle x, y\rangle) = 0\}] .$$

The scope of the quantifier $(\exists \alpha)$ of (1) is a Σ_{2n}^1 relation. Therefore by Moschovakis' Theorem, there exists a Δ_{2n}^1 element α_0 satisfying the Σ_{2n}^1 relation. So R can be order-preservingly embedded into the Δ_{2n}^1 well-ordering given by α_0. Thus we have $\omega(\Pi_{2n}^1) \leq (\Delta_{2n}^1)$ and hence $\omega(\Pi_{2n}^1) = \omega(\Delta_{2n}^1)$.

Theorems 1 and 2 imply the main Theorem.

Appendix

1) We shall give a proof of the following theorem due to Addison-Moschovakis [1; (XI)]:

THEOREM. *APD implies* $\omega(\Delta_k^1) < \omega(E_k^1)$ *for* $k > 0$.

Proof. Let W_k and G_w be as before. We define $B(\Delta_k^1)$ (the set of all Gödel numbers of Δ_k^1 well-orderings) as follows:

$$e \in B(\Delta_k^1) \leftrightarrow e = \langle e_0, e_1\rangle \wedge e_1 \in W_k \wedge [\{e\}^{G_{e_1}} \text{ is a toal function}] \wedge \mathrm{Bord}(\{e_0\}^{G_{e_1}}) .$$

Under the condition $e \in W_k$, $\{e_0\}^{G_{e_1}}$ is Δ_k^1 and hence $B(\Delta_k^1)$ is E_k^1. Let P be:

$$P(m, n) \leftrightarrow m, n \in B(\Delta_k^1) \wedge [\,|\{m_0\}^{G_{m_1}}| < |\{n_0\}^{G_{n_1}}| \\ \vee \{\,|\{m_0\}^{G_{m_1}}| = |\{n_0\}^{G_{n_1}}| \wedge m \leq n\}]\ ,$$

where $m = \langle m_0, m_1\rangle$ and $n = \langle n_1, n_0\rangle$. Obviously P is an E_k^1 well-ordering of natural numbers and $\omega(\Delta_k^1) \leq |P|$. Therefore $\omega(\Delta_k^1) < \omega(E_k^1)$.

2) Addison-Moschovakis [1; (VIII), (IX)] has shown that APD implies $\Delta_k^1 \cap N^N$ is E_k^1 for $k>0$ and it is not Υ_k^1 if k is odd. On the contrary, we can show that $V=L$ implies $\Delta_k^1 \cap N^N$ is Σ_k^1 but not Π_k^1 for all $k>2$. Proof uses Lemma 2 of [6].

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